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Asymptotic analysis for cracks in a catalytic monolith combustor

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Abstract

An asymptotic model is presented for the crack–micro-crack interaction in a material characterized by a low resistance to shear. The material mentioned is obtained by homogenization of the discrete periodic structure used in the design of a catalytic monolith combustor. The results of the asymptotic analysis enable one to compute the stress-intensity factors and predict the shape of the crack trajectory.

1. Introduction

In this work we analyse the model of fracture for a homogenized domain associated with a catalytic monolith combustor studied earlier by Kolaczkowski et al. (1998). A detailed description of catalytic and non-catalytic combustion reactions that take place in a catalytic monolith is available in Hayes and Kolaczkowski (1997). In this paper, we consider crack propagation in a ceramic monolith with square shaped cells. A cross-section of the monolith is illustrated in Fig. 1. The cellular monolith structure may be formed by extrusion of material in the form of a paste through a die, and then firing at high temperatures to form the ceramic.

In a catalytic combustion application, combustible gaseous species flow in a stream of air in the channels, which are coated with a thin layer of wash-coat that contains the catalyst. It is on the surface of the walls (in the layer of wash-coat) where, the catalytic combustion takes place. As described in Hayes and Kolaczkowski (1997) p. 383, channels in a monolith reactor may not operate in an identical manner. For example, in the event of a fuel maldistribution across the face of the monolith, higher

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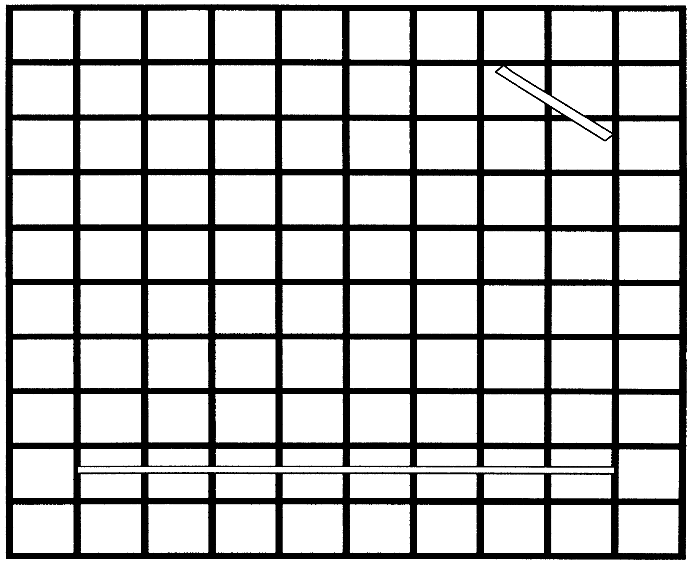

Fig. 1. Cracks in a lattice.

concentrations of fuel could lead to higher rates of reaction and hence, higher temperatures. This would result in variations in temperature across the cross-section of the monolith which as a result of thermal stresses could cause a crack to form. The monolith structure is considered to be a homogenized material which is anisotropic and for certain types of lattices it has a low resistance to shear. We present the asymptotic formulae for effective elastic moduli to high-order accuracy, and these formulae take into account the geometry of the periodic structure as well as the shape of junctions between the elements of the structure. A crack is considered in a homogenized orthotropic material with a small shear modulus and it interacts with a small micro-crack of arbitrary orientation. Here we develop an accurate asymptotic procedure for evaluation of the stress-intensity factors and prediction of the shape of the crack propagating in a catalytic monolith combustor containing micro-cracks.

Asymptotic approach to the analysis of the two-dimensional elastic interaction between the main crack and micro-cracks was implemented by Hori and Nemat-Nasser (1987) and Gong and Horii (1989) for the case of isotropic media. Assuming that the distance between the main crack and a micro-crack is large compared to the length of the micro-crack, the authors presented the series approximations for the stress-intensity factors and the numerical simulations of local stability of the main crack. For the class of isotropic materials the asymptotic analysis of crack deviation is presented in Cotterell and Rice (1980), Movchan et al. (1998) and the crack–defect interaction accompanied by the crack deviation is discussed in Movchan et al. (1991) and Movchan and Movchan (1995). It is important to mention that the present problem (for a material with a small shear modulus) is singularly perturbed, and analytical models of fracture for this type of media were not published elsewhere.

In Section 2 we present the governing equations and the form of the matrix of effective elastic moduli. Section 3 describes the main steps of the asymptotic algorithm, and the model problems involved are discussed in Section 4. There are three main steps: (i) analysis of the field associated with a large crack; (ii) construction of the dipole field associated with a micro-crack arbitrary oriented in orthotropic media with low resistance to shear; (iii) evaluation of the stress-intensity factors. Asymptotic formulae for the stress-intensity factors are derived in Section 5. The effects of singular perturbation associated with a small shear modulus are discussed in Section 6, where we consider the case of small area fraction for the composite media. Numerical results are presented for the perturbation of the stress-intensity factors due to interaction of the large crack with a micro-crack. The results of computations show the presence of a high-gradient region in a strip containing a micro-crack. The results of the asymptotic analysis enable us to proceed further and predict the shape of the crack propagating in a catalytic monolith combustor with micro-defects. Asymptotic formulae and results of numerical simulations for the crack trajectories are included in Section 7.

2. Formulation of the problem

Let **u** be the displacement field which satisfies the boundary value problem

$$\mathscr{D}^{\mathrm{T}}\left(\frac{\partial}{\partial x}\right)\mathscr{H}\mathscr{D}\left(\frac{\partial}{\partial x}\right)\mathbf{u}=0 \quad \text{in } \mathbb{R}^2\backslash\left\{\mathbf{M}\bigcup m_\varepsilon\right\},$$

$$\mathscr{D}^{\mathrm{T}}(\mathbf{n})\mathscr{H}\mathscr{D}\left(\frac{\partial}{\partial x}\right)\mathbf{u}=\mathbf{p}(\mathbf{x}) \quad \text{on } M^{\pm}\bigcup m_\varepsilon^{\pm}, \tag{2.1}$$

where M, m_ε are the cracks specified by

$$M=\left\{\mathbf{x}\colon |x_1|< a,\, x_2=0\right\},$$

$$m_\varepsilon=\left\{\mathbf{x}\colon x_j=x_j^0+l_jt,\, |t|<\varepsilon\right\}, \tag{2.2}$$

and x_j^0, l_j are constants (see Fig. 2). The second crack m_ε has a small length defined by a non-dimensional parameter $0<\varepsilon<1$; for the sake of convenience we assume that $l_1=\cos\beta$, $l_2=\sin\beta$, where β characterises the orientation of the crack m_ε. The differential operator $\mathscr{D}$ is given as

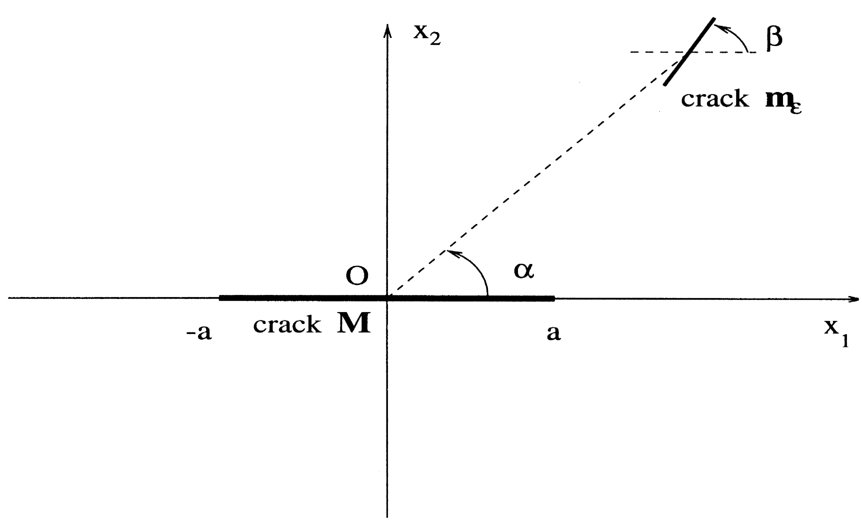

Fig. 2. Geometry of the problem: a macro-crack M and a micro-crack m_ε.

$$\mathscr{D}\left(\frac{\partial}{\partial x}\right) = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} & \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} \end{bmatrix} \tag{2.3}$$

and the positive definite matrix $\mathscr{H}$ of elastic coefficients has the block-diagonal form

$$\mathscr{H} = \begin{pmatrix} c_{11} & fc_{12} & 0 \\ fc_{12} & c_{22} & 0 \\ 0 & 0 & f^2 c_{33} \end{pmatrix}, \quad 0 < f \ll 1; \tag{2.4}$$

so that the material is orthotropic and the shear modulus is small; the quantities c_{ij} have the same order-of-magnitude. The vector **n** denotes the unit outward normal on the crack faces, and the smooth traction **p** is supposed to be self-balanced, i.e.

$$\int_{\smile_\pm (M^\pm \smile m_\varepsilon^\pm)} p_j \, \mathrm{d}s = 0, \quad j = 1, 2; \quad \int_{\smile_\pm (M^\pm \smile m_\varepsilon^\pm)} (x_2 p_1 - x_1 p_2) \, \mathrm{d}s = 0. \tag{2.5}$$

We seek the solution **u** which vanishes at infinity.

The above formulation involves two small parameters ε and f, where the first one defines the normalized size of the crack m_ε, and the second parameter characterises the resistance of elastic medium to shear.

Our objective is to develop an accurate asymptotic algorithm for description of the field **u** and the stress-intensity factors at the ends of the crack M. Finally, our intention is to show that this algorithm enables one to predict the trajectory of a crack propagation in a non-homogeneous low-shear-modulus medium.

The medium is a monolith with square shaped cells as illustrated in Fig. 1. The channels are small (e.g. 1×1 mm) and the wall thickness is about 0.1 mm. The matrix $\mathscr{H}$ gives the components of the homogenized elastic moduli, and it has the form

$$\mathscr{H} = \frac{Qf}{2}\begin{bmatrix} 1 + d_1 f & d_2 f & 0 \\ d_2 f & 1 + d_1 f & 0 \\ 0 & 0 & \frac{f^2}{4}(1 + d_3 f) \end{bmatrix}, \quad Q = \frac{4\mu(\lambda + \mu)}{2\mu + \lambda} \tag{2.6}$$

where f is the area fraction for the region occupied by the elastic material with the Lamé elastic moduli λ and μ. The leading-order terms in (2.6) were discussed in Kolaczkowski et al. (1998). The constant coefficients d_1, d_2 and d_3 depend upon the shape of the junction region. For a particular case of the junction shown in Fig. 1 and for Poisson's ratio $\nu = 0.3$ we have $d_1 = 0.933$, $d_2 = -0.229$ and $d_3 = 1.196$. Details of the derivation are given in Appendix B.

3. Main steps of the asymptotic algorithm

The asymptotic approximation of the displacement field **u** has the form

$$\mathbf{u} \sim \mathbf{v}(\mathbf{x}; f) + \mathbf{w}(\xi, \varepsilon, f), \tag{3.1}$$

where $\mathbf{v}$ is the field associated with the elastic plane containing a single crack M; the boundary layer field $\mathbf{w}$ depends on the scaled coordinates

$$\xi_j = \frac{x_j - x_j^0}{\varepsilon}, \quad j = 1, 2, \tag{3.2}$$

and it is introduced to compensate for an error produced by the field $\mathbf{v}$ in the boundary conditions (2.1) imposed on $m_\varepsilon^\pm$. The field $\mathbf{w}$ depends on both small parameters f and ε (this dependence is unknown a priori), and we aim to specify the asymptotic formula for $\mathbf{w}$ when $|\xi| \to \infty$.

Finally, the resulting field $\mathbf{u}$ yields non-zero tractions on $M^\pm$ and hence, non-zero stress intensity factors, which are different from ones associated with the field $\mathbf{v}$.

The asymptotic study involves three main steps:

1. Analysis of a single-crack problem and specification of the stress field $\sigma_{ij}(\mathbf{v})$.
2. The boundary layer analysis for a small crack m_ε.
3. Derivation of asymptotic formulae for the stress intensity factors; applications to modelling of a quasistatic crack propagation.

4. Model problems

4.1. Singular integral equations describing a crack in orthotropic media

Let ϕ_1, ϕ_2 be Hölder's functions on $(-a, a)$, except its ends, have at the points $\xi = a$ and $\xi = -a$ an integrable singularity

$$\phi_j(\xi) = O\left\{(a \mp \xi)^{-1/2}\right\}, \quad \xi \to \pm a \mp 0$$

and satisfy the orthogonality condition

$$\int_{-a}^{a} \phi_j(\xi)\, \mathrm{d}\xi = 0. \tag{4.1}$$

Then the following integral equations with the Cauchy kernel

$$\frac{\lambda_j}{\pi} \int_{-a}^{a} \frac{\phi_j(\xi)}{\xi - x_1}\, \mathrm{d}\xi = f_j(x_1), \quad |x_1| < a \quad (j = 1, 2) \tag{4.2}$$

have a unique solution; here we assume that f_j are smooth and λ_j are real constants which depend on the elastic moduli of the media.

In the physical model, the functions ϕ_j describe the derivatives of the displacement jumps across the crack. The functions f_j represent the components of tractions. Namely, if $\mathbf{u} = (u_1, u_2)$ is the displacement vector then

$$\phi_j(x_1) = \frac{\partial u_j}{\partial x_1}(x_1, +0) - \frac{\partial u_j}{\partial x_1}(x_1, -0), \quad \operatorname{supp} \phi_j \in [-a, a] \tag{4.3}$$

and

$$f_j(x_1) = \sigma_{j2}(x_1, +0) = \sigma_{j2}(x_1, -0), \quad j = 1, 2, \tag{4.4}$$

where σ_{ij} are stress components associated with the displacement $\mathbf{u}$. The constants λ_j are defined by $\lambda_1 = \lambda_{21}$, $\lambda_2 = \lambda_{12}$ where λ_{21}, λ_{12} are introduced in Appendix A [formulae (A.23), $\gamma = 0$]. The solution of eqns (4.2) under the condition (4.1) is given by the Keldysh–Sedov formula (Sedov, 1972)

$$\phi_j(x_1) = -\frac{1}{\pi\lambda_j\sqrt{a^2 - x_1^2}} \int_{-a}^{a} \frac{\sqrt{a^2 - \xi^2} f_j(\xi)}{\xi - x_1}\, \mathrm{d}\xi. \tag{4.5}$$

The stress components exhibit the square root singularity at the ends of the crack, and the stress-intensity factors are defined by

$$K_{\mathrm{I}}^{\pm} = \lim_{x_1 \to \pm a \pm 0} \sqrt{2\pi|x_1 \mp a|}\,\sigma_{22}(x_1, 0),$$

$$K_{\mathrm{II}}^{\pm} = \lim_{x_1 \to \pm a \pm 0} \sqrt{2\pi|x_1 \mp a|}\,\sigma_{12}(x_1, 0). \tag{4.6}$$

The explicit formulae for the stress-intensity factors are

$$K_{\mathrm{I}}^{\pm} = -\frac{1}{\sqrt{\pi a}} \int_{-a}^{a} f_2(\xi) \left(\frac{a + \xi}{a - \xi}\right)^{\pm 1/2} \mathrm{d}\xi,$$

$$K_{\mathrm{II}}^{\pm} = -\frac{1}{\sqrt{\pi a}} \int_{-a}^{a} f_1(\xi) \left(\frac{a + \xi}{a - \xi}\right)^{\pm 1/2} \mathrm{d}\xi. \tag{4.7}$$

4.2. Dipole field associated with a crack in orthotropic media

Let $\mathbf{u}$ be the displacement field which satisfies the boundary value problem

$$c_{11}\frac{\partial^2 u_1}{\partial x_1^2} + fc_{12}\frac{\partial^2 u_2}{\partial x_1\, \partial x_2} + \frac{1}{2} f^2 c_{33} \frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0,$$

$$c_{22}\frac{\partial^2 u_2}{\partial x_2^2} + fc_{12}\frac{\partial^2 u_1}{\partial x_1\, \partial x_2} + \frac{1}{2} f^2 c_{33} \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0, \quad \mathbf{x} \in \mathbb{R}^2 \backslash \{\mathrm{m}\}, \tag{4.8}$$

and

$$n_1 \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \end{pmatrix} + n_2 \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} = \mathbf{p}, \quad \mathbf{x} \in m^{\pm}, \tag{4.9}$$

$$\|\mathbf{u}\| \to 0 \quad \text{as} \quad \|\mathbf{x}\| \to \infty. \tag{4.10}$$

Here $\mathbf{n} = (n_1, n_2)$ is the unit outward normal on the crack surface, and $\mathbf{p}$ is a constant vector of tractions. The stress components σ_{ij} are specified by

$$\sigma_{11} = c_{11}\frac{\partial u_1}{\partial x_1} + fc_{12}\frac{\partial u_2}{\partial x_2}, \tag{4.11}$$

$$\sigma_{12} = \frac{1}{2} f^2 c_{33}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right), \tag{4.12}$$

$$\sigma_{22} = c_{22}\frac{\partial u_2}{\partial x_2} + fc_{12}\frac{\partial u_1}{\partial x_1}. \tag{4.13}$$

The crack m is defined as $m = \{\mathbf{x}: x_j = l_j t, |t| < 1\}$, $l_1 = \cos\beta$, $l_2 = -\sin\beta$ (see Fig. 2). The problem (4.8)–(4.10) is solvable and its solution is presented in Appendix A ($\gamma = \beta$). Here we note that the stress components in the plane with the crack are equal to

$$\sigma_{mn}(\eta_1, \eta_2) = \frac{1}{\alpha_{22}}\frac{\partial}{\partial \eta_1}\sum_{k=1}^{2} g_k \operatorname{Im}\sum_{j=1}^{2}\frac{(-1)^{m+n+k+j}\mu^{m+n+k-4}}{\nu_j}$$

$$\times\left[\left(\eta_1 + \frac{\eta_2}{\mu_j}\right) - \left(\eta_1 + 1 + \frac{\eta_2}{\mu_j}\right)^{1/2}\left(\eta_1 - 1 + \frac{\eta_2}{\mu_j}\right)^{1/2}\right], \tag{4.14}$$

where (η_1, η_2) are local coordinates associated with the crack

$$\eta_1 = x_1\cos\beta + x_2\sin\beta,$$

$$\eta_2 = -x_1\sin\beta + x_2\cos\beta, \tag{4.15}$$

and

$$\left|\arg\left(\eta_1 \pm 1 + \frac{\eta_2}{\mu_j}\right)\right| < \pi. \tag{4.16}$$

The constants α_{22}, g_k, μ_j and ν_j are specified by formulae (A.28), (A.26), (A.17) and (A.19) in Appendix A ($\gamma = \beta$).

As $\|\boldsymbol{\eta}\| \to \infty$ the stress components have the following asymptotic behaviour

$$\sigma_{mn}(\eta_1, \eta_2) = -\frac{1}{2\alpha_{22}}\sum_{k=1}^{2} g_k \operatorname{Im}\sum_{j=1}^{2}\left[\frac{(-1)^{m+n+k+j}\mu^{m+n+k-4}}{\nu_j\left(\eta_1 + \mu_j^{-1}\eta_2\right)^2} + O\left\{\left(\eta_1 + \frac{\eta_2}{\mu_j}\right)^{-3}\right\}\right]. \tag{4.17}$$

Formula (4.17) shows that at infinity we deal with the dipole field; namely it decays with the same rate as the first-order derivatives of components of Green's tensor.

5. Asymptotics of the stress-intensity factors

(1) First, consider the crack M whose faces $M^{\pm}$ are loaded by given tractions and assume that these tractions are self-balanced (the principal force and moment vectors are equal to zero). It corresponds to the limit case (when $\varepsilon \to 0$) of the problem (2.1) when we neglect the presence of the small crack m_ε. Let

$\mathbf{v}$ denote the displacement field. Then $\phi_1 = [\partial v_1/\partial x_1]$, $\phi_2 = [\partial v_2/\partial x_1]$ are specified by (4.5), where $f_i = p_j(x_1)$ are given tractions, and λ_1, λ_2 are replaced by $\lambda_{21}, \lambda_{12}$ defined in (A.23). The stress components $\sigma_{ij}(\mathbf{v}; \mathbf{x})$ are defined by (A.20), where $\gamma = 0$. We introduce the notation

$$\sigma_{ij}^0 = \sigma_{ij}\big(\mathbf{v}; \mathbf{x}^0\big) \tag{5.1}$$

for the leading terms of stress components evaluated in the vicinity of the small crack m_ε. Here, we assume that tractions on $m_\varepsilon^\pm$ vanish. The field $\mathbf{v}$ produces an error in the traction boundary condition on the faces $m_\varepsilon^\pm$:

$$\sum_j \sigma_{ij}^0 n_j. \tag{5.2}$$

(2) In order to compensate for the above error we introduce a boundary layer, i.e. a function $\mathbf{w}(\xi)$ specified in scales coordinates

$$\xi = \frac{\mathbf{x} - \mathbf{x}^0}{\varepsilon}, \tag{5.3}$$

as a solution of the following boundary value problem

$$\mathscr{D}^{\mathrm{T}}\left(\frac{\partial}{\partial \xi}\right)\mathscr{H}\mathscr{D}\left(\frac{\partial}{\partial \xi}\right)\mathbf{w}(\xi) = 0 \quad \text{in } \mathbb{R}^2\backslash\mathrm{m},$$

$$\mathscr{D}^{\mathrm{T}}(\mathbf{n})\mathscr{H}\mathscr{D}\left(\frac{\partial}{\partial \xi}\right)\mathbf{w}(\xi) = -\boldsymbol{\sigma}^0 \cdot \mathbf{n} \quad \text{on } m^\pm,$$

$$\mathbf{w}(\xi) \to 0 \quad \text{as } \|\xi\| \to \infty. \tag{5.4}$$

Here m is the scaled crack

$$m = \big\{\xi\colon \xi_j = l_j t, |t| < 1\big\}; \tag{5.5}$$

l_j are the same constants as in (2.2). The field

$$\mathbf{v}(\mathbf{x}) + \varepsilon\mathbf{w}(\xi) \tag{5.6}$$

satisfies, to order $O(\varepsilon)$ the homogeneous traction boundary condition on $m_\varepsilon^\pm$. Namely,

$$\sigma_i^{(n)}\big(\mathbf{v}(\mathbf{x}) + \varepsilon\mathbf{w}(\xi); \mathbf{x}\big) = \sigma_i^{0(n)} + \big(\mathbf{x} - \mathbf{x}^0\big) \cdot \nabla\sigma_i^{(n)}\big(\mathbf{v}; \mathbf{x}^0\big) + \sigma_i^{(n)}(\mathbf{w}; \xi) + O\big(\|\mathbf{x} - \mathbf{x}^0\|^2\big), \quad \mathbf{x} \in m_\varepsilon. \tag{5.7}$$

Equivalently,

$$\sigma_i^{(n)}\big(\mathbf{v}(\mathbf{x}) + \varepsilon\mathbf{w}(\xi); \mathbf{x}\big) = \varepsilon\xi \cdot \nabla\sigma_i^{(n)}\big(\mathbf{v}; \mathbf{x}^0\big) + O(\varepsilon^2), \quad \mathbf{x} \in m_\varepsilon. \tag{5.8}$$

In the formulae above we adopted the notation

$$\boldsymbol{\sigma}^{(n)} = n_1\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix} + n_2\begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix}, \tag{5.9}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit outward normal. In the local rotated coordinates

$$\boldsymbol{\eta} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \xi \tag{5.10}$$

the stress components have the asymptotics (4.17) at infinity where $\gamma = \beta$, λ_{ij}, g_j are specified by formulae (A.23) and (A.26), and f_1, f_2 are defined as

$$f_1 = f_1^* = \tfrac{1}{2}\sin 2\beta\left(\sigma_{11}^0 - \sigma_{22}^0\right) - \cos 2\beta\sigma_{12}^0,$$

$$f_2 = f_2^* = -\sin^2\beta\sigma_{11}^0 - \cos^2\beta\sigma_{22}^0 + \sin 2\beta\sigma_{12}^0. \tag{5.11}$$

(3) The field $\varepsilon\mathbf{w}(\xi)$ produces a discrepancy in the traction boundary conditions (2.1) on the faces $M^\pm$ of the 'large' crack. The stress components $\sigma_{ij}(\varepsilon\mathbf{w};\mathbf{x})$ evaluated on $M^\pm$ have the order $O(\varepsilon^2)$ [see (4.17)]. In order to compensate for this error we introduce a field $\varepsilon^2\mathbf{q}(\mathbf{x})$ such that

$$\mathscr{D}^{\mathrm{T}}\left(\frac{\partial}{\partial x}\right)\mathscr{H}\mathscr{D}\left(\frac{\partial}{\partial x}\right)\mathbf{q}(\mathbf{x}) = 0 \quad \text{in } \mathbb{R}^2\backslash\mathbf{M},$$

$$\mathscr{D}^{\mathrm{T}}(\mathbf{n})\mathscr{H}\mathscr{D}\left(\frac{\partial}{\partial x}\right)\mathbf{q}(\mathbf{x}) = \pm\mathbf{F}(\mathbf{x}) \quad \text{on } M^\pm, \tag{5.12}$$

where

$$F_1 = \tfrac{1}{2}\sin 2\beta\left[\Sigma_{22}\left(x_1', x_2'\right) - \Sigma_{11}\left(x_1', x_2'\right)\right] - \cos 2\beta\Sigma_{12}\left(x_1', x_2'\right),$$

$$F_2 = -\sin^2\beta\Sigma_{11}\left(x_1', x_2'\right) - \cos^2\beta\Sigma_{22}\left(x_1', x_2'\right) - \sin 2\beta\Sigma_{12}\left(x_1', x_2'\right), \tag{5.13}$$

$$x_1' = \cos\beta\left(x_1 - x_1^0\right) - \sin\beta x_2^0,$$

$$x_2' = -\sin\beta\left(x_1 - x_1^0\right) - \cos\beta x_2^0, \tag{5.14}$$

and

$$\Sigma_{mn}(\eta_1, \eta_2) = -\frac{1}{2\alpha_{22}^*}\sum_{k=1}^{2} g_k^* \,\mathrm{Im}\sum_{j=1}^{2}\frac{(-1)^{m+n+k+j}\mu_j^{*m+n+k-4}}{v_j^*\left(\eta_1 + \dfrac{\eta_2}{\mu_j^*}\right)^2}, \tag{5.15}$$

$$g_1^* = \frac{\lambda_{22}^* f_2^* - \lambda_{12}^* f_1^*}{\delta_0^*}, \quad g_2^* = \frac{-\lambda_{21}^* f_2^* + \lambda_{11}^* f_1^*}{\delta_0^*}. \tag{5.16}$$

The expressions for the quantities λ_{mn}^*, δ_0^*, α_{22}^*, μ_j^*, v_j^* are obtained from (A.23), (A.27), (A.8), (A.17) and (A.19) by replacing γ by the value β. Also we assume that the field $\mathbf{q}(\mathbf{x})$ decays at infinity.

On this step the approximation for the displacement vector becomes

$$\mathbf{u}(\mathbf{x}, \varepsilon) \sim \mathbf{v}(\mathbf{x}) + \varepsilon\mathbf{w}(\boldsymbol{\eta}) + \varepsilon^2\mathbf{q}(\mathbf{x}). \tag{5.17}$$

As $x_1 \to \pm a$ and $x_2 = 0$, the stress components σ_{j2} are specified by

$$\sigma_{12} = \frac{1}{\sqrt{2\pi|x_1 \mp a|}}\left[K_{\rm II}^{\pm}(\mathbf{v}) + \varepsilon^2 K_{\rm II}^{\pm}(\mathbf{q})\right] + \text{smaller terms,} \tag{5.18}$$

$$\sigma_{22} = \frac{1}{\sqrt{2\pi|x_1 \mp a|}}\left[K_{\rm I}^{\pm}(\mathbf{v}) + \varepsilon^2 K_{\rm I}^{\pm}(\mathbf{q})\right] + \text{smaller terms.} \tag{5.19}$$

The correction terms $\varepsilon^2 K_{\rm I}^{\pm}(\mathbf{q})$, $\varepsilon^2 K_{II}^{\pm}(\mathbf{q})$ associated with the field $\mathbf{q}$ have the form

$$K_m^{\pm}(\mathbf{q}) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\sum_{n=1}^{2}\lambda_{mn}\,\mathrm{Im}\sum_{j=1}^{2}E_{nj}\frac{1}{h_{j\mp}}\left(\frac{h_{j-}^{1/2}}{h_{j+}^{1/2}} - \frac{h_{j+}^{1/2}}{h_{j-}^{1/2}}\right), \quad \arg h_{j\pm} \in (-\pi, \pi), \tag{5.20}$$

where $K_m^{\pm}$ is equal to $K_{\rm I}^{\pm}$ for $m = 1$, and $K_{\rm II}^{\pm}$ for $m = 2$; the quantities E_{nj} are defined by formulae

$$E_{nj} = \frac{1}{2\delta_0\alpha_{22}^* v_j^* \rho_{j+}^2}\sum_{k=1}^{2} g_k^*(-1)^{k+j}\mu_j^{*k}\sum_{m=1}^{3}\Lambda_{jnm} \quad (n, j = 1, 2), \tag{5.21}$$

$$\Lambda_{j11} = \frac{1}{\mu_j^{*2}}\left(\lambda_{22}\sin^2\beta - \frac{1}{2}\lambda_{12}\sin 2\beta\right), \quad \Lambda_{j12} = \frac{1}{\mu_j^*}(-\lambda_{22}\sin 2\beta + \lambda_{12}\cos 2\beta),$$

$$\Lambda_{j13} = \lambda_{22}\cos^2\beta + \frac{1}{2}\lambda_{12}\sin 2\beta, \quad \Lambda_{j21} = \frac{1}{\mu_j^{*2}}\left(-\lambda_{21}\sin^2\beta + \frac{1}{2}\lambda_{11}\sin 2\beta\right),$$

$$\Lambda_{j22} = \frac{1}{\mu_j^*}(\lambda_{21}\sin 2\beta - \lambda_{11}\cos 2\beta), \quad \Lambda_{j23} = -\lambda_{21}\cos^2\beta - \frac{1}{2}\lambda_{11}\sin 2\beta. \tag{5.22}$$

The quantities $h_{j\pm}$, ρ_{j+} depend on the position of the small crack and the material properties

$$h_{j\pm} = x_1^0 \pm a + \frac{x_2^0}{\rho_j}, \quad \rho_j = \frac{\rho_{j+}}{\rho_{j-}}, \tag{5.23}$$

$$\rho_{j+} = \cos\beta - \frac{\sin\beta}{\mu_j}, \quad \rho_{j-} = \sin\beta + \frac{\cos\beta}{\mu_j}. \tag{5.24}$$

6. Asymptotics for small area fraction *f*

In this section we analyse the case when the parameter *f* introduced in (2.6) is small, i.e., the elastic medium exhibits low resistance to shear.

The compliance matrix $\mathscr{H}^{-1}$ has the form

$$\mathscr{H}^{-1} = \frac{2}{Q}\begin{bmatrix} \dfrac{1+d_1 f}{f\delta_*} & -\dfrac{d_2}{\delta_*} & 0 \\ -\dfrac{d_2}{\delta_*} & \dfrac{1+d_1 f}{f\delta_*} & 0 \\ 0 & 0 & \dfrac{4}{f^3(1+d_3 f)} \end{bmatrix}, \tag{6.1}$$

where $\delta_* = (1+d_1 f)^2 - d_2^2 f^2$. The quantities α_{ij} [see (A.8)] are defined by

$$\alpha_{11} = \alpha_{22} = \frac{2}{Q}\left(\frac{1+d_1 f}{f\delta_*} - \frac{1}{2}d_f \sin^2 2\beta\right)$$

$$\alpha_{66} = \frac{2}{Q}\left[\frac{8\cos^2 2\beta}{f^3(1+d_3 f)} + \frac{2\sin^2 2\beta}{f(1+d_1 f - d_2 f)}\right],$$

$$\alpha_{12} = \frac{2}{Q}\left(-\frac{d_2}{\delta_*} + \frac{\sin^2 2\beta}{2}d_f\right),$$

$$\alpha_{16} = -\alpha_{26} = \frac{d_f}{Q}\sin 4\beta, \tag{6.2}$$

where

$$d_f = \frac{1+d_1 f}{f\delta_*} + \frac{d_2}{\delta_*} - \frac{4}{f^3(1+d_3 f)}. \tag{6.3}$$

Characteristic polynomial. For the case $\beta = 0, \pi/2$ eqn (A.17) is written as

$$f^2 z^4 + \frac{8}{1+d_1 f}\left(\frac{\delta_*}{1+d_3 f} - \frac{d_2 f^3}{4}\right)z^2 + f^2 = 0. \tag{6.4}$$

Note that the equation degenerates (it reduces its order) as $f \to 0$. The solutions of (6.4) are given by $z_1 = \mu_1, z_2 = \mu_2, z_3 = -\mu_1, z_4 = -\mu_2$ where $\mathscr{R}\mu_j = 0$ $(j = 1, 2)$ and

$$\mu_j = \frac{2i}{f}R_f^{1/2}\left[1 + (-1)^j\left(1 - \frac{f^4}{16R_f^2}\right)^{1/2}\right]^{1/2}, \quad R_f = \frac{4\delta_* - d_2 f^3(1+d_3 f)}{4(1+d_1 f)(1+d_3 f)}. \tag{6.5}$$

As $f \to 0$,

$$\mu_1 = \frac{if}{2\sqrt{2}}\left(1 - \frac{d_0 f}{2}\right) + O\left(f^3\right), \quad \mu_2 = \frac{2\sqrt{2}i}{f}\left(1 + \frac{d_0 f}{2}\right) + O(f). \tag{6.6}$$

So $\mu_1 \to 0$ and $\mu_2 \to \infty$ as $f \to 0$; here $d_0 = d_1 - d_3$. Thus, two roots z_1 and z_3 are getting small, whereas the moduli $|z_2| = |z_4|$ are large. Accordingly, the quantities ν_1, ν_2 from (A.19) and λ_{ij} [see (A.23)] are approximated by the formulae

$$\nu_1 = \frac{4\sqrt{2}i}{f}\left[1 + \frac{d_0}{2}f + O\left(f^2\right)\right],$$

$$\nu_2 = \frac{32\sqrt{2}i}{f^3}\left[1 + \frac{3d_0}{2}f + O\left(f^2\right)\right] \tag{6.7}$$

and

$$\lambda_{11} = \lambda_{22} = 0, \quad \lambda_{jk} = \frac{Q\sqrt{2}}{16} f^2\left[1 + \frac{d_0}{2}f + O\left(f^2\right)\right], \quad k, j = 1, 2; \quad k \neq j. \tag{6.8}$$

Formulae (A.26) imply

$$g_j(x_1) = \frac{16}{Q\sqrt{2f^2}}\left[1 - \frac{d_0}{2}f + O\left(f^2\right)\right] f_j(x_1), \quad j = 1, 2. \tag{6.9}$$

Examples:

(1) *The case when the crack m_ε is parallel to $M(\beta = 0)$*. Assume that the crack faces $M^\pm$ are loaded by constant tractions $\pm(f_1, f_2)$ and that the crack m_ε is free of tractions. The integral equations (4.1) reduce to

$$\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x_1}\int_{-a}^{a}\frac{\Phi_j(\xi)}{\xi - x_1}\,\mathrm{d}\xi = \frac{8\sqrt{2}}{Qf^2}\left[1 - \frac{d_0}{2}f + O\left(f^2\right)\right] f_j, \tag{6.10}$$

and hence, the displacement jumps across the crack M grow as f tends to 0, i.e.

$$\Phi_j(\xi) = -\frac{8\sqrt{2}}{Q}f_j\left[\frac{1}{f^2} - \frac{d_0}{2f} + O(1)\right]\sqrt{a^2 - \xi^2}. \tag{6.11}$$

It reflects the fact that the problem is singularly perturbed with respect to small f. In the text below we consider expansions in ε assuming that f is fixed. The stress-intensity factors $K_{\mathrm{I}}^{0,\pm}$, $K_{\mathrm{II}}^{0,\pm}$ are finite and given by formulae (4.7) with $f_j, j = 1, 2$, being constants:

$$K_{\mathrm{I}}^{0,\pm} = -f_2\sqrt{\pi a}, \quad K_{\mathrm{II}}^{0,\pm} = -f_1\sqrt{\pi a}. \tag{6.12}$$

The unperturbed stress components σ_{ij}^0 associated with the crack M have the form

$$\sigma_{mn}^0\left(x_1^0, x_2^0\right) = \frac{1}{\alpha_{22}}\sum_{k=1}^{2} g_k \,\mathrm{Im}\sum_{j=1}^{2}\frac{(-1)^{m+n+k+j}\mu_j^{m+n+k-4}}{\nu_j}N_j, \tag{6.13}$$

where

$$N_j = 1 - \frac{1}{2}\left(\frac{H_{j+}^{1/2}}{H_{j-}^{1/2}} + \frac{H_{j-}^{1/2}}{H_{j+}^{1/2}}\right). \tag{6.14}$$

The quantities $H_{j\pm}$ are defined as follows

$$H_{j\pm} = x_1^0 + \frac{x_2^0}{\mu_j} \pm a, \quad |\arg H_{j\pm}| < \pi. \tag{6.15}$$

As $f \to 0$, the coefficients N_1, N_2 have the following asymptotics

$$N_1 = \frac{f^2 a^2}{16(x_2^0)^2}\left[1 - \frac{if}{\sqrt{2x_2^0}}\left(x_1^0 - i\sqrt{2}d_0 x_2^0\right) + O(f^2)\right], \tag{6.16}$$

$$N_2 = 1 - m_1 - i\,\mathrm{sgn}\,x_1^0 m_2 f\left(1 - \frac{d_0}{2}f\right) + c_1 f^2 + O(f^3), \quad |x_1^0| > a, \tag{6.17}$$

$$N_2 = 1 - i\,\mathrm{sgn}\,x_1^0 m_1 + m_2 f\left(1 - \frac{d_0}{2}f\right) + ic_2 f^2 + O(f^3), \quad |x_1^0| < a, \tag{6.18}$$

where

$$m_1 = \frac{|x_1^0|}{\sqrt{|(x_1^0)^2 - a^2|}}, \quad m_2 = \frac{x_2^0 a^2}{2\sqrt{2}|(x_1^0)^2 - a^2|^{3/2}}, \tag{6.19}$$

c_1, c_2 are real constants; due to (6.6) and (6.15) the branch-cuts are chosen in such a way that $(x_1^0 \pm a)^{1/2} = -i|x_1^0 \pm a|^{1/2}$ when $x_1^0 \pm a < 0$.

Remark:. The formulae (6.17) and (6.18) show that the function N_2 expanded in f has singular terms as $|x_1^0| \to a$. The expansion in f gives reliable results outside a neighbourhood of the strip $|x_1^0| < a + \varepsilon_1$ and when $|x_2^0| > \varepsilon_2$ where ε_1, ε_2 are small positive constants. It illustrates the effect of the boundary layer that occurs for small f—the original boundary value problem is singularly perturbed as $f \to 0$.

When $|x_1^0| > a$ formulae (6.13), (6.16) and (6.17) yield as $f \to 0$

$$\sigma_{11}^0(x_1^0, x_2^0) = -\frac{f^2(1 - d_0 f)}{8}\left(1 - m_1 - \frac{a^2}{2(x_2^0)^2}\right)f_2 + O(f^4).$$

$$\sigma_{22}^0(x_1^0, x_2^0) = (1 - m_1)f_2 + O(f^2),$$

$$\sigma_{12}^0(x_1^0, x_2^0) = \frac{f^2(1 - d_0 f)}{2}\left[\left(\frac{a^2}{2(x_2^0)^2} - 1 + m_1\right)\frac{f_1}{4} + \frac{m_2\,\mathrm{sgn}\,x_1^0}{\sqrt{2}}f_2\right] + O(f^4). \tag{6.20}$$

Then, the correction terms for the stress-intensity factors at the ends of the crack M are specified by formulae (5.20), (5.16) and (5.11) where the components σ_{ij}^0 are given by (6.20). The answer for the correction terms for the stress-intensity factors is simplified to the form

$$\Delta K_{\mathrm{I}}^{\pm} = \frac{\varepsilon^2\sqrt{\pi a}\,\mathrm{sgn}\,x_1^0(1 - m_1)}{2(x_1^0 \mp a)\sqrt{(x_1^0)^2 - a^2}}f_2 + O(f^2)$$

$$\Delta K_{\mathrm{II}}^{\pm}=-\frac{\varepsilon^2 f^2\sqrt{\pi a}\,\operatorname{sgn} x_1^0(1-m_1)\left(2x_1^0\pm a\right)x_2^0}{16\left(x_1^0\mp a\right)\left[\left(x_1^0\right)^2-a^2\right]^{3/2}}\left(1-\frac{d_0}{2}\frac{3x_1^0\pm a}{2x_1^0\pm a}f\right)f_2+O\left(f^4\right). \tag{6.21}$$

(2) *The case when the crack m_ε is perpendicular to M ($\beta=\pi/2$)*. The asymptotic procedure used for this case is similar to the one outlined above. When the small crack m_ε is located outside the strip $|x_1|<a$ we derive the following asymptotic formulae for the perturbations of the stress-intensity factors

$$\Delta K_{\mathrm{I}}^{\pm}=-\frac{\varepsilon^2 f^4}{16}\sqrt{\pi a}\left[m_3P_1+\frac{x_2^0\left(2x_1^0\pm a\right)\operatorname{sgn} x_1^0}{\left(x_1^0\mp a\right)\left[\left(x_1^0\right)^2-a^2\right]^{3/2}}P_2\right]+O\left(f^5\right),$$

$$\Delta K_{\mathrm{II}}^{\pm}=-\frac{\varepsilon^2 f^4}{16}\sqrt{\pi a}\left[m_3P_2+\frac{\left(2x_1^0\mp a\right)}{\left(x_2^0\right)^3}P_1\right]+\left(f^5\right). \tag{6.22}$$

Here, $x_2^0\neq 0$, $|x_1^0|>a$, the quantities m_1, m_2 are defined in (6.19), and m_3 as well as P_1, P_2 are given by

$$m_3=\frac{\operatorname{sgn} x_1^0}{\left(x_1^0\mp a\right)\sqrt{\left(x_1^0\right)^2-a^2}}+\frac{1}{\left(x_2^0\right)^2},$$

$$P_1=-\frac{f_2}{8}\left(1-m_1-\frac{a^2}{2\left(x_2^0\right)^2}\right),$$

$$P_2=\left(\frac{a^2}{16\left(x_2^0\right)^2}-\frac{1-m_1}{8}\right)f_1+\frac{f_2m_2\operatorname{sgn}\ x_1^0}{2\sqrt{2}}. \tag{6.23}$$

We remark that the above asymptotic formulae can be used outside neighbourhoods of the strip $|x_1^0|<a$ and the x_1-axis. Also, one can see that the orientation of the crack m_ε is important—in the present case ($\beta=\pi/2$) the correction terms are of order $\varepsilon^2 f^4$ which is smaller compared to the previous case ($\beta=0$). This simple illustration shows the qualitative difference between the crack–defect interaction in the isotropic medium, and in the medium which is orthotropic, with a small shear modulus.

7. Perturbation of the crack trajectory

In this section we consider an example involving a semi-infinite crack

$$M^{\infty}(X)=\{\mathbf{x}\colon x_2=0,\ x_1<X\}$$

interacting with a small crack m_ε described in (2.2). We assume that the displacement $\mathbf{u}$ satisfies the equilibrium system (2.1) and homogeneous traction boundary conditions

$$\mathscr{D}^{\mathrm{T}}(\mathbf{n})\mathscr{H}\mathscr{D}\left(\frac{\partial}{\partial x}\right)\mathbf{u}=0\quad \text{on } M^{\infty,\pm}\bigcup m_\varepsilon^{\pm}, \tag{7.1}$$

where $M^{\infty,\pm}$, $m_\varepsilon^\pm$ denote the crack faces. At infinity, we assume (see Sih and Liebowitz, 1968) that

$$\sigma_{mn} \sim \frac{K_\mathrm{I}^{(0)}}{\sqrt{2\pi r}} \Psi_{mn}(\theta), \quad r \to \infty \quad (m, n = 1, 2), \tag{7.2}$$

where

$$x_1^0 - X = r \cos\theta, \quad x_2^0 = r \sin\theta,$$

$$\Psi_{mn}(\theta) = (-1)^{m+n} \Re \left\{ \frac{1}{\mu_2 - \mu_1} \left[\frac{\mu_2^{m+n-3}}{\left(\cos\theta + \mu_2^{-1} \sin\theta\right)^{1/2}} - \frac{\mu_1^{m+n-3}}{\left(\cos\theta + \mu_1^{-1} \sin\theta\right)^{1/2}} \right] \right\}. \tag{7.3}$$

Due to the interaction with a small crack, the stress-intensity factor K_II at the end of M^∞ will not be zero. However, if we replace $M^\infty(X)$ by a crack $M_\varepsilon^\infty(X)$.

$$M_\varepsilon^\infty(X) = \{\mathrm{x}\colon x_2 = \varepsilon^2 h(x_1), x_1 < X\},$$

where h is smooth then, by appropriate choice of h, we can achieve that $K_\mathrm{II} = 0$ i.e. M_ε^∞ will 'propagate' as a Mode I crack. The latter problem is singularly perturbed, and for the isotropic case the asymptotic algorithm is described in detail by Movchan et al. (1998). Here, we present the summary of results for the case of orthotropic medium. First, the stress-intensity factor K_II is given by

$$\varepsilon^{-2} K_\mathrm{II} = q h'(X) K_\mathrm{I}^{(0)} + \int_{-\infty}^{X} \zeta(x_1) F_1(x_1, X)\, \mathrm{d}x_1 = 0, \tag{7.4}$$

where $\zeta(x_1)$ is the Mode II weight function for a semi-infinite crack

$$\zeta(x_1) = \sqrt{\frac{2}{\pi(X - x_1)}}, \tag{7.5}$$

and F_1 is the first component of the applied load produced by the small crack m_ε

$$F_1(x_1, X) = \lambda_{21}\, \mathrm{Im} \sum_{j=1}^{2} \frac{E_{1j}(X)}{\left(x_1 - x_1^0 - x_2^0 \rho_j^{-1}\right)^2}. \tag{7.6}$$

The coefficient q in (7.4) is equal to 1/2 (the same as in the isotropic case) due to the fact that the microcrack is located on the symmetry axis of the orthotropic two-dimensional medium. In general, for an arbitrary orientation of the crack this coefficient should be replaced by

$$q = 1 + \frac{1}{2} \Re \frac{1}{\mu_1 \mu_2}. \tag{7.7}$$

Due to formulae (5.21) we have

$$E_{ij}(X) = -\frac{K_\mathrm{I}^{(0)}}{\sqrt{2\pi}\lambda_{21}} \sum_{k=1}^{2} g_k^*(X) \Omega_{kj}, \tag{7.8}$$

$$\Omega_{kj} = \frac{(-1)^{k+j}(\mu_j^*)^k}{2\alpha_{11}^* v_j^* \rho_{j+}^2}\left(-\frac{\sin 2\beta}{2\mu_j^{*2}} + \frac{\cos 2\beta}{\mu_j^*} + \frac{\sin 2\beta}{2}\right), \tag{7.9}$$

where $g_k^*(X)$ are defined by (5.16) and

$$f_1^*(X) = -\sin^2\beta\Psi_{11}^*(X) - \cos^2\beta\Psi_{22}^*(X) + \sin 2\beta\Psi_{12}^*(X),$$

$$f_2^*(X) = \frac{1}{2}\sin 2\beta\left[\Psi_{11}^*(X) - \Psi_{22}^*(X)\right] - \cos 2\beta\Psi_{12}^*(X), \tag{7.10}$$

$$\Psi_{mn}^*(X) = (-1)^{m+n}\Re\left\{\frac{1}{\mu_2-\mu_1}\left[\frac{\mu_2^{m+n-3}}{\left(x_1^0 - X + \mu_2^{-1}x_2^0\right)^{1/2}} - \frac{\mu_1^{m+n-3}}{\left(x_1^0 - X + \mu_1^{-1}x_2^0\right)^{1/2}}\right]\right\}. \tag{7.11}$$

The crack trajectory is defined from differential equation (7.4)

$$h(X) = \operatorname{Im}\sum_{j=1}^{2}\sum_{k=1}^{2}\Omega_{kj}\int_{-\infty}^{X} g_k^*(X)\left(x_1^0 + \frac{x_2^0}{\rho_j} - X\right)^{-3/2}\mathrm{d}X,\ \arg\left(x_1^0 + \frac{x_2^0}{\rho_j} - X\right) \in (-\pi, \pi). \tag{7.12}$$

The results of calculations show that the deflection at infinity is the same as the one at the point $x_1 = x_1^0$.

8. Numerical results

Here, we present numerical illustration of the asymptotic formulae derived in the sections above. The tractions components ($\sigma_{22} = -1$, $\sigma_{12} = 0$) are specified on the faces of the macro-crack, and the micro-crack is assumed to be free of fractions. First, we give examples of calculations for the stress-intensity factors. Second, we analyse the trajectory of the crack interacting with a micro-crack. Finally, a local stability is studied for a large crack in an inhomogeneous solid.

(1) Evaluation of the stress-intensity factors. In the numerical calculations we used formula (5.20) for the stress-intensity factors. We remark that the asymptotic formulae (6.21) and (6.22) give good agreement with (5.20) when the micro-crack is located outside a neighbourhood of the strip $|x_1| < a$.

In Figs. 3 and 4, the normalized correction terms for the stress-intensity factors are presented for two parallel cracks interacting with each other. The distance of the micro-crack to the origin is fixed ($R/a = 3$, and the angle $\alpha = \tan^{-1}(x_2^0/x_1^0)$ characterises the position of the micro-crack. The graphs in Figs. 3 and 4 show the presence of the gradient region in the vicinity of the vertical strip containing the micro-crack, and it is shown that the amplitude of oscillation of the stress-intensity factors gets larger for smaller values of the volume fraction parameter f. It is noticeable that the Mode I stress-intensity factor changes sign, and in the text below we analyse this effect in terms of local stability of the macro-crack.

In Figs. 5 and 6 the normalized perturbation terms of the stress-intensity factors are given for the cases when $\alpha = \pi/2$ and $\cos^{-1}a/R + \pi/20$ when the orientation of the small micro-crack changes. The graphs show that there is a point of maximum $\beta^* \neq 0$ for the functions $K_j(\beta)$, and this point depends on α. Fig. 6 corresponds to the case when the micro-crack is located in a neighbourhood of the vertical strip containing the micro-crack (we observe large amplitude of oscillations of the stress-intensity factors for this position of the micro-crack).

(2) The crack trajectory. Here, we use formula (7.12) for the normalized crack deflection h. In Fig. 7

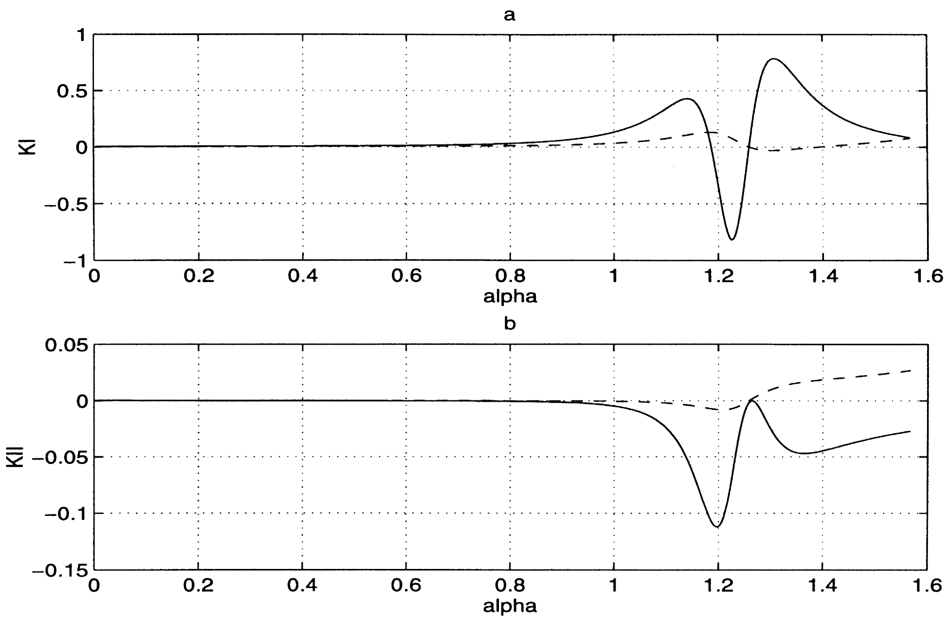

Fig. 3. Corrections in stress-intensity factors vs. α for $\beta = 0$, $f = 0.2$. —right end, - - - left end. (a) The Mode I stress-intensity factors. (b) The Mode II stress-intensity factors.

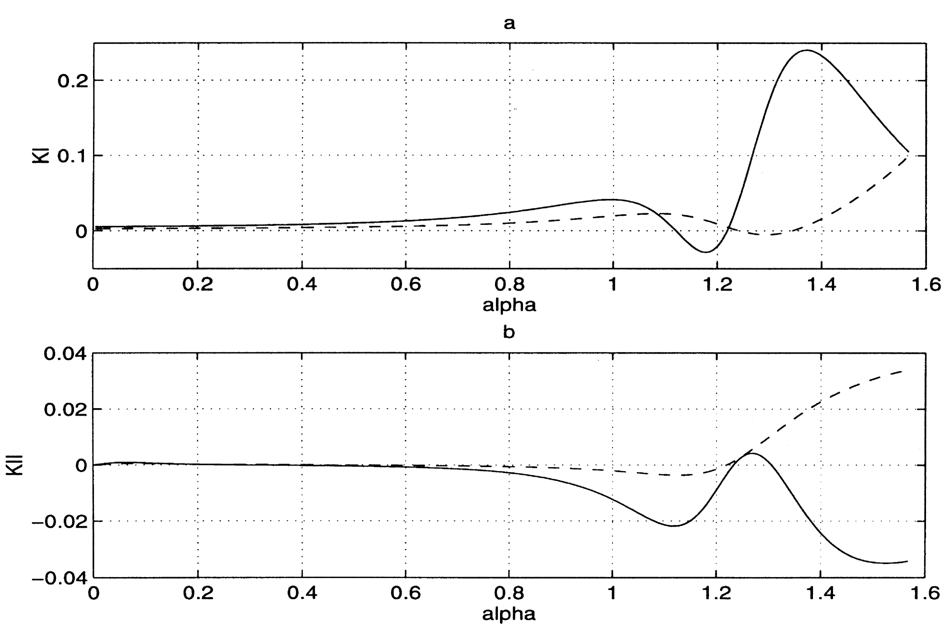

Fig. 4. Correction in stress-intensity factors vs. α for $\beta = 0, f = 0.5$. — right end, - - - left end. (a) The Mode I stress-intensity factors. (b) The Mode II stress-intensity factors.

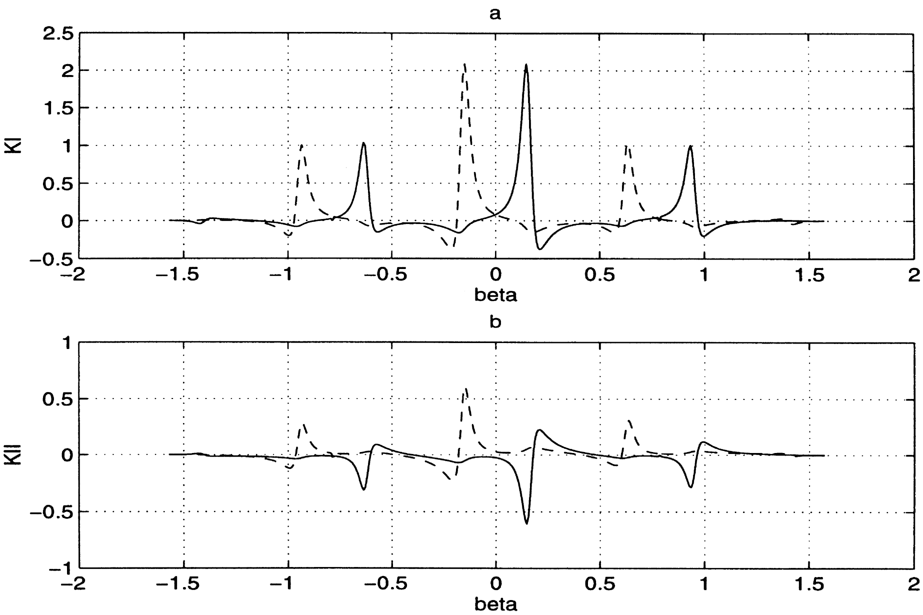

Fig. 5. Corrections in stress-intensity factors vs. β for $\alpha = (\pi/2), f = 0.2$. —right end, - - - left end. (a) The Mode I stress-intensity factors. (b) The Mode II stress-intensity factors.

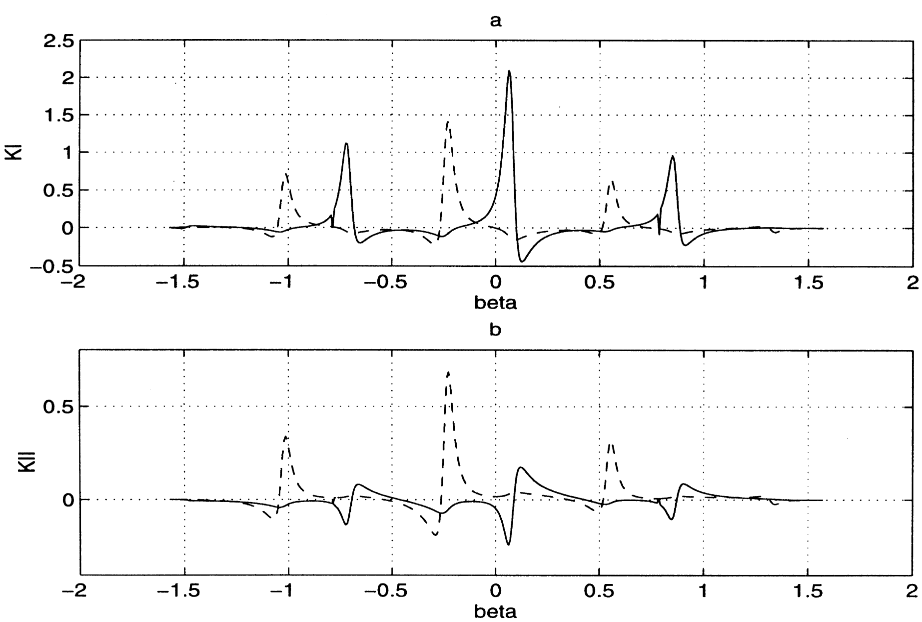

Fig. 6. Corrections in stress-intensity factors vs. β for $\alpha \cos^{-1}(a/R) + (\pi/20), f = 1.2$. — right end, - - - left end. (a) The Mode I stress-intensity factors. (b) The Mode II stress-intensity factors.

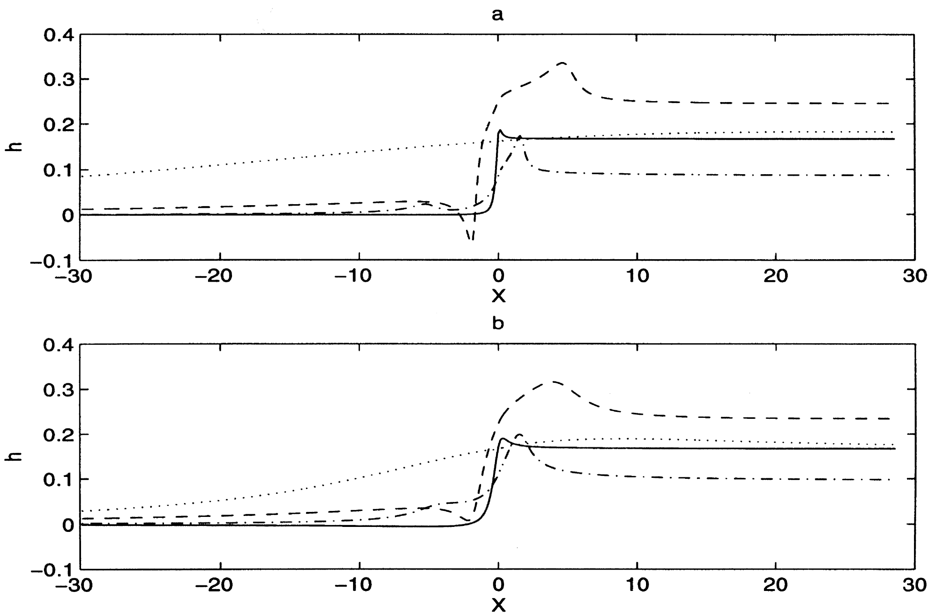

Fig. 7. Normalized crack deflection h. — $\beta = 0$, - - - $\beta = (\pi/6)$, · - · $\beta = (\pi/3)$, ... $\beta = (\pi/2)$. (a) $f = 0.2$. (b) $f = 0.5$.

we show the quantity h for the cases of different orientations of the micro-crack. We note that, in contrast with the case of isotropic material, the deflection of the macro-crack at infinity depends on the angle β. This dependence is shown in Fig. 8 for different values of the volume fraction parameter f. It is observed that for smaller f we have larger positive deflection of the macro-crack.

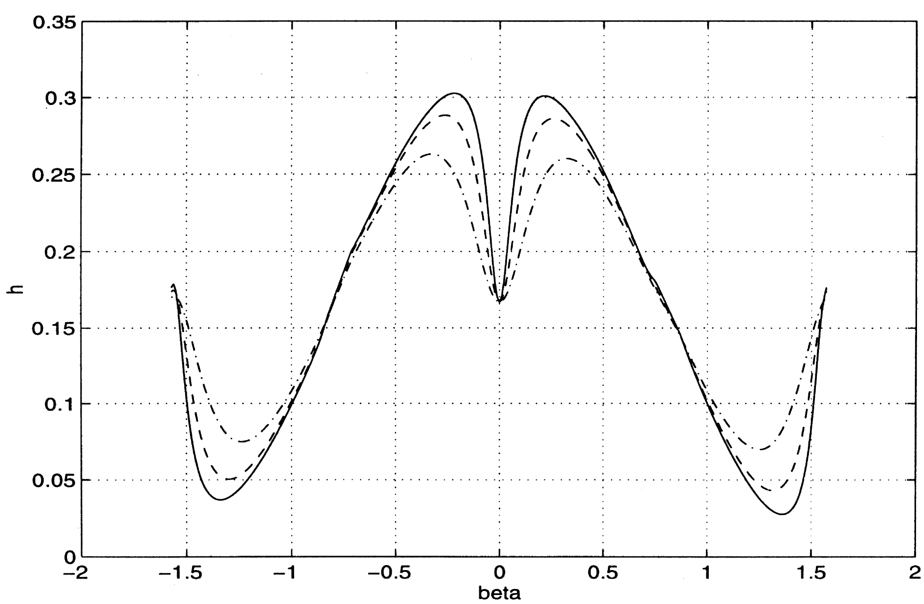

Fig. 8. Dependence of the crack deflection $h(\infty)$ on β.— f = 0.2. - - - f = 0.3. · - · f = 0.5.

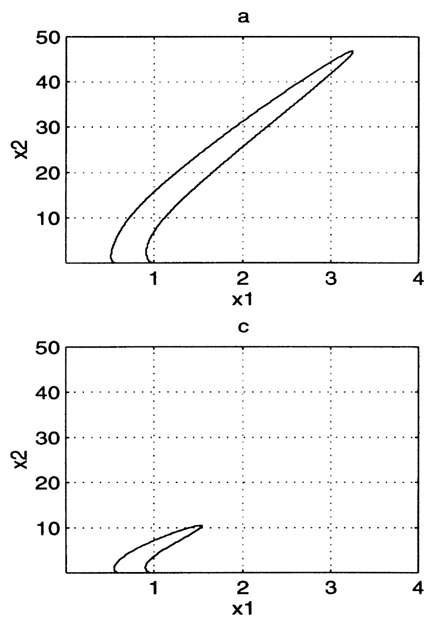

Fig. 9. Regions of local stability for $\beta = 0$, $(R/a) = 3$: (a) $K_{\mathrm{I}}(-a)$ for $f = 0.2$; (b) $K_{\mathrm{I}}(a)$ for $f = 0.2$; (c) $K_{\mathrm{I}}(-a)$ for $f = 0.5$; (d) $K_{\mathrm{I}}(a)$ for $f = 0.5$.

(3) Local stability. As mentioned above, for certain positions of the micro-crack the perturbation in the Mode I stress-intensity factor is negative (the macro-crack is locally stable). Again, we use formula (5.20). In Fig. 9 we show the regions of local stability (regions, bounded by the contours, show location of the micro-crack) for the case of the macro-crack along the segment $[-1, 1]$, $\beta = 0$ and different value of f. Note, that the size of the regions of local stability increases for smaller values of f.

9. Conclusion

We have analysed a challenging singularly perturbed problem of the crack–defect interaction in orthotropic media with a low resistance to shear. Explicit asymptotic formulae for the stress-intensity factors and for the crack deflection have been derived, and the phenomenon of local stability was studied for a macro-crack propagating in inhomogeneous orthotropic solid. The homogenization procedure was applied to obtain the high-accuracy asymptotic formulae for the effective moduli of the lattice structure used in the design of a catalytic monolith combustor. The asymptotic algorithm presented is straightforward for numerical implementation and it is much more efficient compared to the standard FEM software. Applications of these results are in the modelling of fracture of monolith combustors in the car industry and gas turbines.

Acknowledgements

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Appendix A. A finite crack in an orthotropic medium with an arbitrary orientation

Crack problems in orthotropic media are well-studied in the literature. We refer, for example, to the work of Sih and Liebowitz (1968). Here, we present, for the purpose of reference, the account of main formulae for the case of a finite crack of an arbitrary orientation in orthotropic media. The integral equations technique is employed to obtain solutions of model problems described in Section 4 of the main text.

Consider a plane-strain problem when the stress components satisfy homogeneous equilibrium equations

$$\sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad i = 1, 2 \quad \text{in } \mathbb{R}^2 \backslash \{\mathbf{x}: |x_1| < a, x_2 + 0\},$$

$$\sigma_{j2}(x_1, \pm 0) = f_1(x_1), \quad |x_1| < a \tag{A.1}$$

and

$$\sigma_{ij} \to 0 \quad \text{as } \|\mathbf{x}\| \to \infty. \tag{A.2}$$

In addition, we assume that the compatibility equation

$$\nabla^2(\sigma_{11} + \sigma_{22}) = 0 \tag{A.3}$$

is satisfied in $\mathbb{R}^2 \backslash \{\mathbf{x}: |x_1| < a, x_2 = 0\}$. The Airy stress function is introduced in such a way that

$$\alpha_{22} \frac{\partial^4 U}{\partial x_1^4} - 2\alpha_{26} \frac{\partial^4 U}{\partial x_1^3 \, \partial x_2} + (2\alpha_{12} + \alpha_{66}) \frac{\partial^4 U}{\partial x_1^2 \, \partial x_2^2} - 2\alpha_{16} \frac{\partial^4 U}{\partial x_1 \, \partial x_2^3} + \alpha_{11} \frac{\partial^4 U}{\partial x_2^4} = 0. \tag{A.4}$$

In terms of the Airy stress function the stress components can be written as follows

$$\sigma_{11} = \frac{\partial^2 U}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 U}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 U}{\partial x_1 \, \partial x_2}, \tag{A.5}$$

$$\frac{\partial u_1}{\partial x_1} = \alpha_{11} \frac{\partial^2 U}{\partial x_2^2} + \alpha_{12} \frac{\partial^2 U}{\partial x_1^2} - \alpha_{16} \frac{\partial^2 U}{\partial x_1 \, \partial x_2},$$

$$\frac{\partial u_2}{\partial x_2} = \alpha_{12} \frac{\partial^2 U}{\partial x_2^2} + \alpha_{22} \frac{\partial^2 U}{\partial x_1^2} - \alpha_{26} \frac{\partial^2 U}{\partial x_1 \, \partial x_2},$$

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \alpha_{16} \frac{\partial^2 U}{\partial x_2^2} + \alpha_{26} \frac{\partial^2 U}{\partial x_1^2} - \alpha_{66} \frac{\partial^2 U}{\partial x_1 \, \partial x_2}, \tag{A.6}$$

where α_{ij} are the elastic moduli. For orthotropic media there are four independent constants. In the system of coordinates whose axes coincide with the principal symmetry axes the matrix of elastic moduli has the block-diagonal form, so that

$$\varepsilon_{11} = a_{11} \, \sigma_{11} + a_{12} \, \sigma_{22},$$

$$\varepsilon_{22} = a_{21}\,\sigma_{11} + a_{22}\,\sigma_{22},$$

$$\varepsilon_{12} = \tfrac{1}{2}\,a_{66}\,\sigma_{12}, \tag{A.7}$$

Then the constants a_{ij} are given by

$$\alpha_{11} = a_{11}\cos^4\gamma + a_{22}\sin^4\gamma + \tfrac{1}{2}\left(a_{12} + \tfrac{1}{2}\,a_{66}\right)\sin^2 2\gamma,$$

$$\alpha_{22} = a_{11}\sin^4\gamma + a_{22}\cos^4\gamma + \tfrac{1}{2}\left(a_{12} + \tfrac{1}{2}\,a_{66}\right)\sin^2 2\gamma,$$

$$\alpha_{66} = (a_{11} - 2a_{12} + a_{22})\sin^2 2\gamma + a_{66}\cos^2 2\gamma,$$

$$\alpha_{12} = \tfrac{1}{4}\,(a_{11} - 2a_{12} + a_{22} - a_{66})\sin^2 2\gamma + a_{12},$$

$$\alpha_{16} = \left[a_{11}\cos^2\gamma - a_{22}\sin^2\gamma - \left(a_{12} + \tfrac{1}{2}\,a_{66}\right)\cos 2\gamma\right]\sin 2\gamma,$$

$$\alpha_{26} = \left[a_{11}\sin^2\gamma - a_{22}\cos^2\gamma + \left(a_{12} + \tfrac{1}{2}\,a_{66}\right)\cos 2\gamma\right]\sin 2\gamma, \tag{A.8}$$

where γ is the angle of rotation of the symmetry axes with respect to the system of coordinates associated with the crack.

Let ϕ_1, ϕ_2 denote the derivatives of the displacement jumps across the crack

$$\phi_j(x_1) = \left[\frac{\partial u_1}{\partial x_1}\right] = \frac{\partial u_j}{\partial x_1}(x_1, +0) - \frac{\partial u_j}{\partial x_1}(x_1, -0), \quad j = 1, 2, \quad \operatorname{supp}\phi_j \subset [-a, a]. \tag{A.9}$$

Due to the continuity of tractions across the crack

$$[U] = \left[\frac{\partial U}{\partial x_2}\right] = 0, \tag{A.10}$$

and due to (A.9), the derivatives $\partial^2 U/\partial x_2^2$, $\partial^3 U/\partial x_2^3$ have jumps specified by

$$\left[\frac{\partial^2 U}{\partial x_2^2}\right] = \frac{1}{\alpha_{11}}\phi_1(x_1),$$

$$\left[\frac{\partial^3 U}{\partial x_2^3}\right] = \frac{2\alpha_{16}}{\alpha_{11}^2}\phi_1'(x_1) - \frac{1}{\alpha_{11}}\phi_2'(x_1). \tag{A.11}$$

Applying the Fourier transform with respect to x_2-variable to eqn (A.4) and taking into account (A.10) and (A.11) we obtain

$$\left[\alpha_{22}\frac{\mathrm{d}^4}{\mathrm{d}x_1^4}+2\alpha_{26}i\beta\frac{\mathrm{d}^3}{\mathrm{d}x_1^3}-(2\alpha_{12}+\alpha_{66})\beta^2\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}-2\alpha_{16}i\beta^3\frac{\mathrm{d}}{\mathrm{d}x_1}+\alpha_{11}\beta^4\right]U_\beta(x_1)$$

$$=-\phi_2'(x_1)-i\beta\phi_1(x_1), \tag{A.12}$$

where

$$U_\beta(x_1)=\int_{-\infty}^{\infty}U(x_1,x_2)\,\mathrm{e}^{i\beta x_2}\,\mathrm{d}x_2. \tag{A.13}$$

Here, we assume that the function U and its derivatives decay at infinity, and the integrals involved exist. Next, we consider the Fourier transform of (A.12) with respect to x_1. We remark that the integrals of functions highly singular at $x=\pm a$ are treated in the sense of generalized functions. As a result we deduce

$$\left[\alpha_{22}\alpha^4-2\alpha_{26}\alpha^3\beta+(2\alpha_{12}+\alpha_{66})\alpha^2\beta^2-2\alpha_{16}\alpha\beta^3+\alpha_{11}\beta^4\right]U_{\beta\alpha}=-i\beta\phi_{1\alpha}+i\alpha\phi_{2\alpha}, \tag{A.14}$$

where

$$U_{\beta\alpha}=\int_{-\infty}^{\infty}U_\beta(x_1)\,\mathrm{e}^{i\alpha x_1}\,\mathrm{d}x_1,$$

$$\phi_{j\alpha}=\int_{-a}^{a}\phi_j(x_1)\,\mathrm{e}^{i\alpha x_1}\,\mathrm{d}x_1. \tag{A.15}$$

Using (A.14) and inverting the Fourier transform $U_{\beta\alpha}$ with respect to α one can derive

$$U_\beta(x_1)=\frac{1}{2\pi\alpha_{22}}\int_{-\infty}^{\infty}\frac{\left(-i\beta\phi_{1\alpha}+i\alpha\phi_{2\alpha}\right)\mathrm{e}^{-i\alpha x_1}\,\mathrm{d}\alpha}{(\alpha-\mu_1\beta)(\alpha-\mu_2\beta)(\alpha-\bar{\mu}_1\beta)(\alpha-\bar{\mu}_2\beta)}$$

$$=-\frac{i\beta}{\alpha_{22}}\int_{-a}^{a}\phi_1(\xi)g_\beta(\xi-x_1)\,\mathrm{d}\xi+\frac{1}{\alpha_{22}}\int_{-a}^{a}\phi_2(\xi)\,\frac{\partial}{\partial\xi}g_\beta(\xi-x_1)\,\mathrm{d}\xi, \tag{A.16}$$

where μ_1, μ_2 and $\bar{\mu}_1$, $\bar{\mu}_2$ are the roots of the characteristic polynomial

$$\alpha_{22}z^4-2\alpha_{26}z^3+(2\alpha_{12}+\alpha_{66})z^2-2\alpha_{16}z+\alpha_{11}=0, \tag{A.17}$$

and

$$g_\beta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\alpha t}\,\mathrm{d}\alpha}{(\alpha-\mu_1\beta)(\alpha-\mu_2\beta)(\alpha-\bar{\mu}_1\beta)(\alpha-\bar{\mu}_2\beta)}$$

$$=\begin{cases} g_\beta^+(t), & \beta t>0\\ \overline{g_\beta^+}(t), & \beta t<0\end{cases},$$

$$g_\beta^+(t)=\frac{i}{|\beta|^3}\left(\frac{1}{\nu_1}\mathrm{e}^{i\mu_1|\beta t|}-\frac{1}{\nu_2}\mathrm{e}^{i\mu_2|\beta t|}\right). \tag{A.18}$$

Here

$$\nu_1 = (\mu_1 - \mu_2)(\mu_1 - \bar{\mu}_1)(\mu_1 - \bar{\mu}_2),$$

$$\nu_2 = (\mu_1 - \mu_2)(\mu_2 - \bar{\mu}_1)(\mu_2 - \bar{\mu}_2). \tag{A.19}$$

It is assumed that $\mu_1 \neq \mu_2$; otherwise we have to deal with the case of isotropic media. Inverting the transform with respect to β and using formulae (A.5) we obtain the stress components in the form

$$\sigma_{mn}(x_1, x_2) = \frac{1}{\pi\alpha_{22}} \sum_{k=1}^{2} \int_{-a}^{a} \mathrm{Im}\left\{ \frac{(-\mu_1)^{m+n+k-3}}{\nu_1[\mu_1(\xi - x_1) - x_2]} - \frac{(-\mu_2)^{m+n+k-3}}{\nu_2[\mu_2(\xi - x_1) - x_2]} \right\} \phi_k(\xi)\, \mathrm{d}\xi. \tag{A.20}$$

In particular, when (x_1, x_2) does not belong to the crack surface the following representation may be useful

$$\sigma_{mn}(x_1,x_2) = -\frac{1}{\pi\alpha_{22}} \sum_{k=1}^{2} \int_{-a}^{a} \mathrm{Im}\left\{ \frac{(-\mu_1)^{m+n+k-2}}{\nu_1[\mu_1(\xi - x_1) - x_2]^2} - \frac{(-\mu_2)^{m+n+k-2}}{\nu_2[\mu_2(\xi - x_1) - x_2]^2} \right\} \Phi_k(\xi)\, \mathrm{d}\xi, \tag{A.21}$$

where Φ_k ($k = 1, 2$) denotes the displacement jump

$$\Phi_k(x_1) = u_k(x_1, +0) - u_k(x_1, -0). \tag{A.22}$$

The boundary conditions (A.1) yield the system of singular integral equations

$$\frac{\lambda_{11}}{\pi} \int_{-a}^{a} \frac{\phi_1(\xi)}{\xi - x_1}\, \mathrm{d}\xi + \frac{\lambda_{12}}{\pi} \int_{-a}^{a} \frac{\phi_2(\xi)}{\xi - x_1}\, \mathrm{d}\xi = f_2(x_1),$$

$$\frac{\lambda_{21}}{\pi} \int_{-a}^{a} \frac{\phi_1(\xi)}{\xi - x_1}\, \mathrm{d}\xi + \frac{\lambda_{22}}{\pi} \int_{-a}^{a} \frac{\phi_2(\xi)}{\xi - x_1}\, \mathrm{d}\xi = f_1(x_1), \quad -a < x_1 < a. \tag{A.22a}$$

Here

$$\lambda_{11} = \lambda_{22} = \frac{1}{\alpha_{22}} \Re\left(\frac{i\mu_2}{\nu_2} - \frac{i\mu_1}{\nu_1} \right),$$

$$\lambda_{12} = \frac{1}{\alpha_{22}} \Re\left(\frac{i\mu_1^2}{\nu_1} - \frac{i\mu_2^2}{\nu_2} \right), \quad \lambda_{21} = \frac{1}{\alpha_{22}} \Re\left(\frac{i}{\nu_1} - \frac{i}{\nu_2} \right). \tag{A.23}$$

The solution of the system (A.22a) which has integrable singularities at the ends $x_1 = \pm a$ of the crack and satisfies the orthogonality condition

$$\int_{-a}^{a} \phi_j(\xi)\, \mathrm{d}\xi = 0 \quad (j = 1, 2), \tag{A.24}$$

has the form

$$\phi_j(x_1) = -\frac{1}{\pi\sqrt{a^2 - x_1^2}} \int_{-a}^{a} \frac{\sqrt{a^2 - \xi^2}\, g_j(\xi)}{\xi - x_1}\, \mathrm{d}\xi \quad (j = 1, 2), \tag{A.25}$$

where

$$g_1(x_1) = \frac{\lambda_{22} f_2(x_1) - \lambda_{12} f_1(x_1)}{\delta_0},$$

$$g_2(x_1) = \frac{-\lambda_{21} f_2(x_1) + \lambda_{11} f_1(x_1)}{\delta_0}, \tag{A.26}$$

$$\delta_0 = \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}. \tag{A.27}$$

Let us consider a particular case when tractions applied on the crack surface are constant:

$$f_j(x_1) = f_j = \text{const} \quad (j = 1, 2). \tag{A.28}$$

Then the quantities g_1, g_2 are also constant. In this case the functions ϕ_j, Φ_j [see formulae (A.9) and (A.22)] have the form

$$\phi_j(x_1) = \frac{x_1 g_j}{\sqrt{a^2 - x_1^2}}, \quad \Phi_j(x_1) = -g_j\sqrt{a^2 - x_1^2}. \tag{A.29}$$

The stress components (A.21) are simplified as

$$\sigma_{mn}(x_1, x_2) = \frac{1}{\pi\alpha_{22}} \frac{\partial}{\partial x_1} \sum_{k=1}^{2} g_k \,\text{Im}\left\{ \sum_{j=1}^{2} \frac{(-\mu_j)^{m+n+k-3}(-1)^j}{\nu_j} J_j(x_1, x_2) \right\}, \tag{A.30}$$

where

$$J_j(x_1, x_2) = \int_{-a}^{a} \frac{\sqrt{a^2 - \xi^2}\,\mathrm{d}\xi}{\mu_j(\xi - x_1) - x_2}. \tag{A.31}$$

By changing the variable of integration

$$\tau = \frac{\xi + a}{2a}$$

we obtain

$$J_j = \frac{2a}{\mu_j} \int_0^1 \frac{\sqrt{\tau(1-\tau)\,\mathrm{d}\tau}}{\tau + z_j}, \tag{A.32}$$

where

$$z_j = -\frac{\mu_j(x_1 + a) + x_2}{2a\mu_j} \notin [-1, 0]. \tag{A.33}$$

The integral J_j is evaluated in the explicit form

$$J_j(x_1, x_2) = \frac{\pi}{\mu_j}\left[-\left(x_1 + \frac{x_2}{\mu_j}\right) + \left(x_1 + a + \frac{x_2}{\mu_j}\right)^{1/2} \left(x_1 - a + \frac{x_2}{\mu_j}\right)^{1/2} \right], \tag{A.34}$$

$$\left|\arg\left(x_1 \pm a + \frac{x_2}{\mu_j}\right)\right| < \pi, \quad j = 1, 2. \tag{A.35}$$

We remark that for the case of real z_j the integral (A.32) can be found in Bateman and Erdelyi (1954). Substituting (A.34) into (A.30) we obtain (4.14).

Appendix B. High-order corrections of the effective moduli

Here, we describe the details of calculations of the constants d_1, d_2 and d_3 in the formula (2.6) for the effective elastic moduli. We consider the juction region shown in Fig. 10, and note that the constants d_j may change as we choose a different geometry of the region of the junction.

A general homogenization procedure

Consider an elastic plane containing a doubly periodic array of inhomogeneities. The case considered in the present paper corresponds to the situation when the cavities (the channels of the catalytic monolith combustor) are closely located to each other.

Let the displacement vector $\mathbf{u} = (u_1, u_2)^t$ satisfy the following system of equilibrium equations

$$\mathscr{D}^t\left(\frac{\partial}{\partial x}\right)\mathbf{H}(\xi)\mathscr{D}\left(\frac{\partial}{\partial x}\right)u(\mathbf{x}, \xi) = 0 \tag{B.1}$$

where $\mathbf{H}(\xi)$ is a 3×3 doubly periodic with the unit period matrix-function, $\mathscr{D}$ is the matrix differential operator defined in (2.3) and $\boldsymbol{\xi} = (\xi_1, \xi_2)$ are the scaled coordinates

$$(\xi_1, \xi_2) = \varepsilon^{-1}(x_1, x_2)$$

Here ε is the period of the array of inhomogeneities.

The problems of homogenization are discussed in great detail in the books by Bakhvalov and

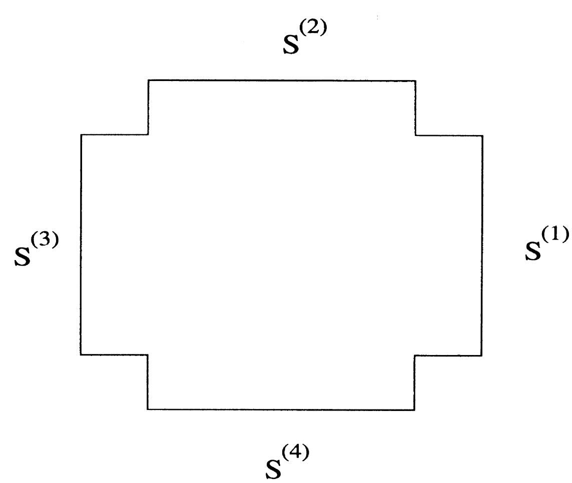

Fig. 10. Junction region.

Panasenko (1989), Bensoussan et al. (1978), Zhikov et al. (1994). Here we present the outline of the asymptotic procedure to explain the derivation of the formula for the effective elastic moduli.

We shall use the following basis in the space of linear displacement fields that produce non-zero stresses

$$\mathbf{V}^{(1)}(\mathbf{x}) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \quad \mathbf{V}^{(2)}(\mathbf{x}) = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \quad \text{and} \quad \mathbf{V}^{(3)}(\mathbf{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}, \tag{B.2}$$

and we also use the vector differential operators $\mathbf{V}^{(j)}(\partial/\partial x),\ j = 1, 2, 3$. Note that

$$\left[\mathbf{V}^{(j)}\left(\frac{\partial}{\partial x}\right)\right]^t \mathbf{V}^{(k)}(x) = \delta_{jk}.$$

The displacement $\mathbf{u}$ is sought in the form of the asymptotic series

$$\mathbf{u} \sim \mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)} + \varepsilon^2 \mathbf{u}^{(2)} + \cdots. \tag{B.3}$$

By direct substitution of (B.3) into (B.1) we derive a recurrent sequence of equations on a unit cell. These equations are supplied with periodicity conditions in ξ, and solvability conditions given as the balance equations for components of the principal force and the principal moment. As a result of this standard procedure, one has that $\mathbf{u}^{(0)}$ is ξ-independent

$$\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x}),$$

and the following representation holds

$$\mathscr{D}^t\left(\frac{\partial}{\partial \xi}\right)\mathbf{H}(\xi)\mathscr{D}\left(\frac{\partial}{\partial \xi}\right)\mathbf{u}^{(1)}(\mathbf{x}, \xi)$$

$$= -\mathscr{D}^t\left(\frac{\partial}{\partial \xi}\right)\mathbf{H}(\xi)\mathscr{D}\left(\frac{\partial}{\partial x}\right)\mathbf{u}^{(0)}(\mathbf{x}) = -\sum_{n=1}^{3} e_n(\mathbf{x})\mathscr{D}^t\left(\frac{\partial}{\partial \xi}\right)\mathbf{H}(\xi)\mathscr{D}\left(\frac{\partial}{\partial x}\right)\mathbf{V}^{(n)}(\mathbf{x}),$$

where e_n are components of the strain vector

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \sqrt{2}\varepsilon_{12} \end{pmatrix} = \mathscr{D}\left(\frac{\partial}{\partial x}\right)\mathbf{u}^{(0)}(\mathbf{x}),$$

and $\mathbf{V}^{(n)}$ are the vectors (B.2). The equilibrium equations for $\mathbf{u}^{(1)}$ are supplied with periodicity conditions, and the solution is sought in the form

$$\mathbf{u}^{(1)}(\mathbf{x}, \xi) = \sum_{n=1}^{3} e_n(\mathbf{x})\mathbf{W}^{(n)}(\xi),$$

where $\mathbf{W}^{(n)}$ are periodic vector functions which satisfy the following system of equilibrium equations

$$\mathscr{D}^t\left(\frac{\partial}{\partial \xi}\right)\mathbf{H}(\xi)\mathscr{D}\left(\frac{\partial}{\partial \xi}\right)\left(\mathbf{W}^{(n)}(\xi) + \mathbf{V}^{(n)}(\xi)\right) = 0$$

on a unit cell. The functions $\mathbf{W}^{(n)}$ compensate for discrepancies produced by $\mathbf{V}^{(n)}$ on the boundaries of inhomogeneities. We also note the standard fact that the problem for $\mathbf{u}^{(1)}$ is solvable. The problem for $\mathbf{u}^{(2)}$ requires the following solvability condition

$$\mathscr{D}^t\left(\frac{\partial}{\partial x}\right)\mathscr{H}\mathscr{D}\left(\frac{\partial}{\partial \xi}\right)u^{(0)}(\mathbf{x}) = 0,$$

where $\mathscr{H} = (\mathscr{H}_{ij})_{i,j=1}^3$

$$\mathscr{H}_{ij} = \frac{1}{2}\sum_{p,q=1}^{3}\int_{\text{unit cell}} \sigma_{pq}\left(\mathbf{V}^{(i)} + \mathbf{W}^{(i)}\varepsilon_{pq}\right)\left(\mathbf{V}^{(j)} + \mathbf{W}^{(j)}\right)\,\mathrm{d}\xi. \tag{B.4}$$

We split the region occupied by the material within the unit cell into five parts: four thin rectangles and the junction region. For the approximation of the fields within thin rectangles we use the asymptotic formulae presented in Kolaczkowski et al. (1998). Here, we add the contribution from the junction region in order to obtain the higher-order approximation of elastic moduli and to evaluate the constants d_1, d_2 and d_3 from (2.6). Assuming that the normalized thickness of rectangles within the unit cell is equal to ε ($0 < \varepsilon \ll 1$) we write the matrix of effective elastic moduli in the form

$$\mathscr{H} = \mathscr{H}^{(1)} + \mathscr{H}^{(2)}, \tag{B.5}$$

where

$$\mathscr{H}^{(1)} = \varepsilon Q\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}, \quad Q = \frac{4\mu(\lambda+\mu)}{2\mu+\lambda}$$

is the part of the matrix of elastic moduli associated with four thin rectangles and

$$\mathscr{H}^{(2)} = \varepsilon^2 Q\begin{pmatrix} \Delta_1 & \Delta_2 & 0 \\ \Delta_2 & \Delta_1 & 0 \\ 0 & 0 & \varepsilon^2\Delta_3 \end{pmatrix}$$

is the contribution from the junction region; the quantities Δ_i, $i = 1, 2, 3$ are unknown constants.

Evaluation of the matrix $\mathscr{H}$

The displacement field in the k-th thin rectangle is approximated in the form (we refer to Kolaczkowski et al., 1998 for detailed calculations),

$$\mathbf{u}(x, t) \sim \mathscr{U}^{(0,k)} + \varepsilon\mathscr{U}^{(1,k)} + \mathscr{V}^{(0,k)} + \varepsilon\mathscr{V}^{(1,k)} + \varepsilon^2\mathscr{V}^{(2,k)} + \varepsilon^3\mathscr{V}^{(3,K)}, \tag{B.6}$$

where

$$\mathscr{U}^{(0,k)} = \begin{pmatrix} A_k x + B_k \\ 0 \end{pmatrix}, \quad \mathscr{U}^{(1,k)} = \begin{pmatrix} 0 \\ -\dfrac{\lambda}{\lambda+2\mu}tA_k \end{pmatrix},$$

$$\mathscr{V}^{(0,k)} = \begin{pmatrix} 0 \\ C_k x^3 + D_k x^2 + E_k x + F_k \end{pmatrix}, \quad \mathscr{V}^{(1,k)} = \begin{pmatrix} -t\left(3C_k x^2 + 2D_k x + E_k\right) \\ 0 \end{pmatrix},$$

$$\mathscr{V}^{(2,k)} = \begin{pmatrix} 0 \\ -\dfrac{\lambda}{\lambda + 2\mu} t^2 (3C_k x + D_k) \end{pmatrix}, \quad \mathscr{V}^{(3,k)} = \begin{pmatrix} \left[\dfrac{3\lambda + 4\mu}{\lambda + 2\mu} t^3 - \dfrac{3(\lambda + \mu)}{\lambda + 2\mu} t\right] C_k \\ 0 \end{pmatrix},$$

where $A_k, B_k, C_k, D_k, E_k, F_k$ are constant coefficients, and (x, t) are local coordinates with the origin at the left of the rectangle and t being a scaled variable such that $|t| < 1/2$ within the rectangle. The field (B.6) satisfies the equilibrium equations in the rectangle and homogeneous traction boundary conditions on the upper and lower surfaces.

As mentioned in (B.1), we have to consider three types of fields $\mathbf{U}^{(j)} = \mathbf{V}^{(j)} + \mathbf{V}^{(j)}$, $j = 1, 2, 3$, where the vectors $\mathbf{V}^{(j)}$ are given in (B.2) and $\mathbf{W}^{(j)}$ are doubly periodic. In evaluation of the energy integral only the constants A_k, C_k and D_k are required. These constants are specified from the periodicity boundary conditions allocated for the exterior end regions of thin rods

$$\boldsymbol{\sigma}^{(n)}(\mathbf{U}^{(j)}) \cdot \mathbf{e}_1^{(k)} = \boldsymbol{\sigma}^{(n)}(\mathbf{V}^{(j)}) \cdot \mathbf{e}_1^{(k)}, \quad \mathbf{U}^{(j)} \cdot \mathbf{e}_2^{(k)} = \mathbf{V}^{(j)} \cdot \mathbf{e}_2^{(k)} = 0 \quad (j = 1, 2),$$

$$\boldsymbol{\sigma}^{(n)}\left(\mathbf{U}^{(3)}\right) \cdot \mathbf{e}_2^{(k)} = \boldsymbol{\sigma}^{(n)}\left(\mathbf{V}^{(3)}\right) \cdot \mathbf{e}_2^{(k)}, \quad \mathbf{U}^{(3)} \cdot \mathbf{e}_1^{(k)} = \mathbf{V}^{(3)} \cdot \mathbf{e}_1^{(k)} = 0, \tag{B.7}$$

where $\boldsymbol{\sigma}^{(n)}$ is the vector of tractions, and $(\mathbf{e}_1^{(k)}, \mathbf{e}_2^{(k)})$ is the local Cartesian basis associated with the k-th rectangle. Analysis of the boundary layer near the right end of the rectangle yields

$$\frac{\partial \mathscr{U}_1^{(0,k)}}{\partial e_1^{(k)}} = \frac{\partial\left(\mathscr{V}^{(j)} \cdot \mathbf{e}_1^{(k)}\right)}{\partial e_1^{(k)}}, \quad j = 1, 2$$

and

$$\frac{\partial \mathscr{V}_2^{(0,k)}}{\partial e_1^{(k)}} = \frac{\partial\left(\mathscr{V}^{(3)} \cdot \mathbf{e}_2^{(k)}\right)}{\partial e_1^{(k)}}, \frac{\partial^2 \mathscr{V}_2^{(0,k)}(x)}{\partial x^2} = 0 \quad \text{at } x = \frac{1}{2}.$$

Consequently, for the fields $\mathbf{V}^{(j)} + \mathbf{W}^{(j)}, j = 1, 2, 3$ the non-zero constants A_k, C_k and D_k are specified in the form

$$j = 1: \quad A_1 = A_3 = 1,$$

$$j = 2: \quad A_2 = A_4 = 1,$$

$$j = 3: \quad C_1 = -C_2 = C_3 = -C_4 = -\sqrt{2}, \quad D_1 = -D_2 = D_3 = -D_4 = \frac{3}{\sqrt{2}}.$$

It is verified directly that in the local basis $(\mathbf{e}_1^{(k)}, \mathbf{e}_2^{(k)})$ associated with the k-th rectangle

$$\sigma_{11}(\mathbf{u}) = Q\left[A_k - 2\varepsilon t(3C_k x + D_k)\right],$$

$$\sigma_{12}(\mathbf{u}) = \frac{3Q\mu(\lambda + \mu)}{\lambda + 2\mu}\varepsilon^2(4t^2 - 1) = O(\varepsilon^2).$$

We note that, as $x = O(\varepsilon)$

$$\sigma_{11}(\mathbf{u}) = Q(A_k - 2\varepsilon t D_k) + O(\varepsilon^2). \tag{B.8}$$

Finally, we consider three traction model problems for the functions $\mathbf{U}^{(j)}, j = 1, 2, 3$ in the junction region shown in Fig. 10.

Problem 1. The stress components satisfy the equilibrium equations and homogeneous traction boundary conditions everywhere except the parts $S^{(1)}$ and $S^{(3)}$ of the boundary where $\sigma_{11} = Q$ the shear stresses vanish on the boundary.

Problem 2 is similar to problem 1. The only change is that we replace $S^{(1)}$, $S^{(3)}$ by $S^{(2)}$, $S^{(4)}$ and assume that $\sigma_{22} = Q$ on $S^{(2)}, S^{(4)}$ and homogeneous traction boundary conditions are satisfied on the remaining part of the boundary.

Problem 3. The stress components satisfy the equilibrium equations, and $\sigma_{11} = -2tD_k$ on $S^{(k)}$, $k = 1$, 3, $\sigma_{22} = -2tD_k$ on $S^{(k)}$, $k = 2$, 4. The shear stresses are equal to zero on the boundary.

In the numerical computations the stress components are normalized by Young's modulus, and we used Poisson's ratio $\nu = 0.3$. The numerical computations were performed with the COSMOS/M Finite Element Software. The constants Δ_1, Δ_2 are specified in the form

$$\Delta_1 = Q\int_{S^{(1)}} u_x \, \mathrm{d}s = 1.3646, \tag{B.9}$$

$$\Delta_2 = Q\int_{S^{(2)}} u_y \, \mathrm{d}s = -0.4571, \tag{B.10}$$

where u_x, u_y are the displacement components shown in Figs. 11 and 12. To evaluate Δ_3 it is sufficient to compute the elastic associated with the model problem 3. The contour plot of the energy is given in Fig. 13. As a result, we have

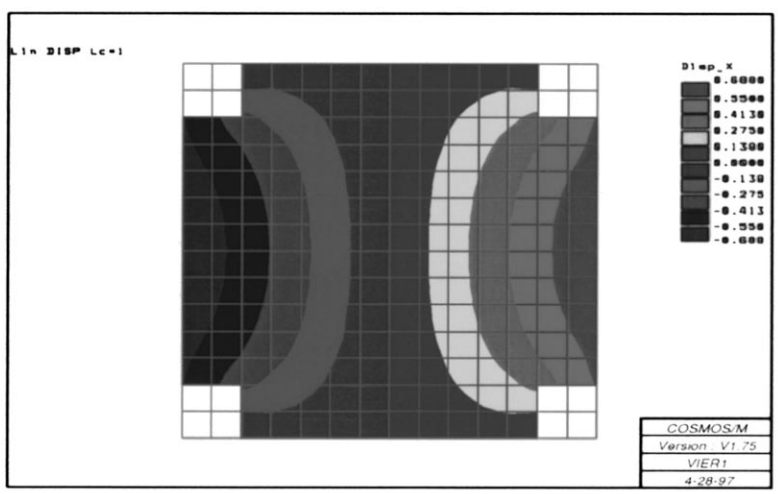

Fig. 11. Displacement u_x in the junction region in the model problem 1 required for evaluation of Δ_1 [formula (B.9)].

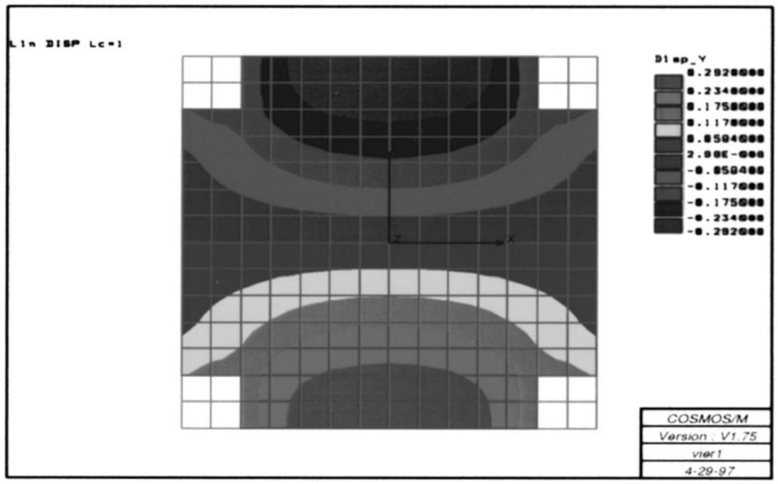

Fig. 12. Displacement u_y in the junction region in the model problem 1 required for evaluation of Δ_2 [formula (B.10)].

$$\Delta_3 = 0.891. \tag{B.11}$$

Finally, we remark that the area fraction f for the region occupied by the material is related to quantity ε by

$$2\varepsilon - \varepsilon^2 = f,$$

and, therefore,

$$\varepsilon = \frac{1}{2} f + \frac{1}{8} f^2 + O\left(f^3\right) \tag{B.12}$$

for small f. Using (B.9)–(B.12) and the representation (B.5) we derive

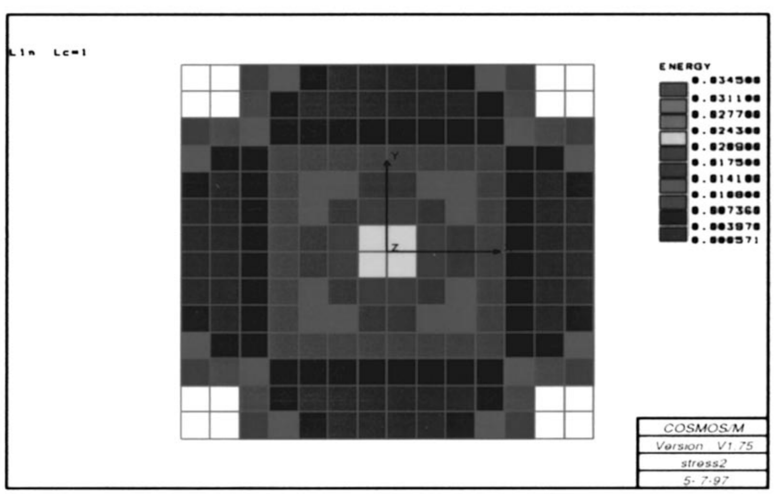

Fig. 13. Energy distribution in the model problem 3 required for evaluation of Δ_3 [formula (B.11)].

$$\mathscr{H} \sim \frac{1}{2} Qf \begin{bmatrix} 1 + d_1 f & d_2 f & 0 \\ d_2 f & 1 + d_1 f & 0 \\ 0 & 0 & \dfrac{f^2}{4}(1 + d_3 f) \end{bmatrix},$$

where

$$d_1 = \frac{1}{4} + \frac{\Delta_1}{2} = 0.933,$$

$$d_2 = \frac{\Delta_2}{2} = -0.229,$$

$$d_3 = \frac{3}{4} + \frac{\Delta_3}{2} = 1.196.$$

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